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# Coordinate System In Quantum Gravity

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## Abstract

We discuss a Gedanken experiment in which we construct a physical coordinate system which covers the universe. Using general properties of quantum gravity we find that the minimum uncertainty of the coordinate system is  $\sqrt[5]{R}$  where  $R$  is the radius of the universe.

The existence of a minimum length at Planck's scale has been proposed from the study of thought experiments which are model independent [1, 2, 5, 4] or through an analysis of collisions at planckian energies in the context of string theories[3, 8] (for recent review see [7]). In [9] we studied the problem of time measurement in quantum gravity. We have shown that one can not synchronize two clocks to an accuracy better than  $\Delta T = \sqrt{8 \log \frac{x}{x_c}}$  (in units where  $\hbar = c = G = 1$ ), where  $x_c$  is the shortest distance for which general relativity is a good approximation to quantum gravity, and  $x$  is the distance between the clocks. The fact that  $\Delta T$  is not just a constant of nature of the order of the Planck scale as one might expect, but an increasing unbounded function of  $x$  indicates that the problem of finding a quantum theory of gravity is more intricate than the problem of renormalization at the Planck scale. The reason that one might not be too concerned about it, is that the distance between the clocks is bounded by the diameter of the universe ( $\approx 10^{60}$ ). Therefore, even if one assumes that  $x_c = 1$  one gets  $\Delta T_{max} \approx 33$ , which is not too far from 1, yet for an open universe  $\Delta T_{max} = \infty$ . In this letter we would like to generalize the two-clock problem mentioned above by studying a physical coordinate system (p.c.s.)<sup>1</sup> that covers the universe. In particular we would like to investigate the dependence of the accuracy of the p.c.s. on the size of the universe.

While in general relativity the clocks and the rods, which are used to construct a p.c.s., must have an infinitesimal energy momentum tensor in order not to change the metric, in quantum theories without gravitation we must use p.c.s. with  $\Delta P_\mu \rightarrow \infty$  in order to get  $\Delta X_\mu \rightarrow 0$ . Although theoretically, there is no limitation on the accuracy of the p.c.s. in quantum theories without gravitation, in practice the energy in the universe is finite. Thus there is a practical limitation on the accuracy of a p.c.s. in quantum theories even when gravitation is absent. In other words— in classical theories we can use

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<sup>1</sup>By p.c.s. we mean any physical device which can measure any event or events which occur at the universe with some accuracy.

infinitesimal energy of the universe in order to construct coordinate system for the whole universe with infinitesimal uncertainty, while in quantum theories even if we use the whole energy in the universe to construct a coordinate system, we still have, due to the uncertainty principle, only a finite accuracy.

In quantum gravity the problem in constructing a p.c.s. is much more intricate, since, as we shall see later on, even theoretically the accuracy of the p.c.s. is bounded from below.

In order to build a p.c.s. with accuracy  $b$ ,<sup>2</sup> which covers the whole universe, one must construct a device which is able to measure the location at which any events in the universe occur with accuracy  $b$  therefore one must divide the universe into 4-dimensional cells with volume  $b^4$ . When one considers quantum theory one finds that due to the uncertainty principle  $\Delta P_\mu \geq b^{-1}$  in each cell. In the weak field approximation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1 \quad (1)$$

we have

$$P_\mu = \int d^3x T_{0\mu} \quad (2)$$

where  $T_{\mu\nu}$  is the energy momentum tensor. Thus

$$\Delta \int_{cell} d^3x T_{0\mu} > b^{-1} \quad (3)$$

in each cell. Note that eq.(3) is not a consequence of vacuum fluctuation in each cell, but a constraint that the energy momentum tensor of the physical clocks and rods must satisfy in order to have accuracy  $b$ .

There is a well know solution to the field equation in the weak field approximation where one works in an harmonic coordinate system (for instance [10])

$$g_{\mu\nu}(x, t) = \eta_{\mu\nu} + 4 \int d^3x' \frac{S_{\mu\nu}(x', t - |x - x'|)}{|x - x'|} \quad (4)$$

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<sup>2</sup>Since we assume the weak field approximation  $b$  is nothing but the bare accuracy of the rods and clocks which are used to construct the p.c.s.

where

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T_{\gamma}^{\gamma}. \quad (5)$$

Since the uncertainties of any two cells are independent of each other we get

$$\Delta^2 g_{\mu\nu}(x, t) = 16 \sum_{cells} \frac{(\Delta \int d^3x, S_{\mu\nu})^2}{(x - x_{cell})^2} \quad (6)$$

$S_{0i} = T_{0i}$  ; thus we have in each cell

$$\Delta \int d^3x S_{0i} \geq b^{-1}. \quad (7)$$

Note that  $\Delta N_{cell} \approx \frac{dV}{b^3}$ . Since  $b \ll R$  , we can replace the sum in eq.(6) by an integral, and use eq.(7) to obtain

$$\Delta^2 g_{0i} = 16 \int_0^\pi d\theta \int_0^{2\pi} d\varphi \int_0^R dr \frac{\sin \theta}{b^5} \quad (8)$$

therefore

$$\Delta g_{0i} \geq \frac{8\sqrt{\pi}}{b^2} \sqrt{\frac{R}{b}}. \quad (9)$$

The invariant distance is  $ds = \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$  , hence the real accuracy of the p.c.s. is not  $b$  but ,

$$\max\{\Delta(\sqrt{g_{\mu\nu}dx^\mu dx^\nu})\} \quad (10)$$

Denoting the p.c.s. accuracy by  $d$  one has :

$$d \geq \max(b, \sqrt[4]{\frac{R}{b}}) \geq \sqrt[5]{R}. \quad (11)$$

Note that although we calculate  $d$  in a particular choice of coordinate  $d$  as defined in eq.(10) is invariante under coordinate transformation, hence the conclusions are independent of the choice of coordinates. The reason why we did not calculate the exact numerical constant in eq.(10) is that the entire calculation was based on the weak field approximation, while for  $b = \sqrt[5]{R}$  we

get from eq.(9)  $\Delta g_{0i} > 1$ . Thus one can consider the final result only as a qualitative result

$$d \propto \sqrt[5]{R}. \quad (12)$$

The meaning of  $\sqrt[5]{R}$  as a lower bound to the accuracy of the p.c.s. is the following : if one would try to construct a p.c.s. using clocks and rods which are more accurate then  $\sqrt[5]{R}$ , than the quantum fluctuation of their accuracy would be larger then the their bare accuracy , and the classical description of the p.c.s. will be violated. Notice that for  $R = 10^{60}$  , the energy density of the p.c.s. is  $R^{-\frac{4}{5}} \approx 10^{-48}$  at the minimum accuracy. While the energy density of the universe is only  $\approx 10^{-123}$ , thus the classical description of the large scale universe is valid. Since the weak field approximation is not valid in the early universe a further investigation is needed in order to find out, when in the early universe the quantum corrections play an essential role and the classic description of cosmology is no longer valid. Still, it is tempting to use the above calculation , which was based on the weak field approximation in order to compare between the accuracy  $d$  and the wave length of the background radiation  $l_{bac} \approx \frac{R}{\sqrt[5]{N}}$ ; we find that  $l_{bac} = d$  at  $R \approx N^{\frac{5}{12}} \approx 10^{33}$  which is approximately the size of the universe at the big bang.

Note that  $\sqrt[5]{R}$  is the local accuracy of the p.c.s. i.e., the minimum invariant distance of each cell. Yet it is not the uncertainty in the invariant distance between any two events in the universe, which is the global accuracy of the p.c.s.. In order to calculate the global accuracy one needs to remember that when a coordinate system is constructed of cells of length  $d \pm \Delta d$ , the uncertainty in the distance between two events which are separated by  $N$  cells is  $d + \sqrt{N}\Delta d$ . In our case  $d = b$  ,  $N_{max} = \frac{R}{b}$  and  $\Delta d = \Delta g_{0i}b$ . Therefore we find that the global accuracy of the p.c.s. is

$$d_{gl} = b + 8\sqrt{\pi}\frac{R}{b^2} \quad (13)$$

The minimum is

$$d_{gl,min} \approx \sqrt[3]{R} \quad (14)$$

Note that at the minimum  $\Delta g \approx \frac{1}{\sqrt[3]{R}}$ , since  $R \gg 1$  the weak field approximation is valid.

While the fact that the global accuracy is a function of a global variable  $R$  is not surprising, the fact that the local accuracy in quantum gravity is, operationally a function of a global variable is surprising and somewhat disturbing, since by definition, coordinate system is a local concept. In our opinion this result gives rise to the question whether it is possible to describe quantum gravity by means of local quantum field theory.

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